Persistent Cohomology and Circle-valued coordinates

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Outline

1. Motivation: Intrinsic coordinates
2. Theory: Persistent cohomology and circle-valued maps
3. Practice: Finding and interpreting parametrizations
Finding coordinates

- Overall goal is to understand pointclouds.
- Data comes with coordinates. Different coordinate choice might concentrate the intrinsic information.
- We want to find few and very relevant intrinsic coordinates. Ideal case: 2d or 3d plots with a clear and relevant geometry.
In order to find few intrinsic coordinates, we want to stick close to the local dimension. Some shapes take up too many coordinates.

Locally 1-dimensional. Globally 2 coordinates needed to describe all points. The shape doesn’t fit in $\mathbb{R}$.

Similar problems arise with sphere and torus.
Suggested fix

Circle-valued coordinates

- Use $S^1 = [0, 1]/(0 \sim 1)$ as additional coordinate space
- Fixes the circle
- Fixes the torus
- Occurs naturally:
  - Phase coordinates for waves
  - Angle coordinates for directions
  - Any recurrent phenomenon
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Problem remains: how do we find circle-valued coordinates?

**Persistent cohomology**

- Degree one cohomology equivalent to circle-valued maps
- Persistence picks out relevant features from noise
- Once a feature-rich parameter has been found, we can work in ordinary (non-persistent) cohomology theories

We compute persistent cohomology by adapting the zig zag persistence algorithm to the dual diagram.
From cohomology to circle-valued parametrizations

We use the natural isomorphism $H^1(X; \mathbb{Z}) \cong [X, S^1]$

**Issues**

- Easy to compute: $H^1(X; \mathbb{Z}/p)$, with coefficients over a small prime. Linear algebra, coefficients fit inside machine word, division in $O(1)$ by lookup tables. Needed for the isomorphism: $H^1(X; \mathbb{Z})$.

We can, as long as $H^2(X; \mathbb{Z})$ has no $p$-torsion, lift $H^1(X; \mathbb{Z}/p) \to H^1(X; \mathbb{Z})$.

- The representative chains for $H^1(X; \mathbb{Z})$ yields very non-smooth maps: sends all data points to 0, and wraps the edges in the complex around the target circle.

We can smooth a cocycle in $C^1(X; \mathbb{Z})$ by moving it to a harmonic cocycle in $C^1(X; \mathbb{R}) \cap C_1(X; \mathbb{R})$ belonging to the same cohomology class in $H^1(X; \mathbb{Z})$. 
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Parametrized circles
Parametrized circles
Parametrized circles
Parametrized circles
Knots and links
Knots and links
Knots and links
Torus
Torus
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Motivation
Theory
Practice

Torus
Torus
Torus

Correlation plot for this torus parametrization
Torus

Correlation plot for a wedge of two circles
Correlation plot for an elliptic curve in $\mathbb{CP}^2$
Pop quiz
Pop quiz
Pop quiz
Pop quiz
Pop quiz
Mumford dataset


4.2 · 10^6 pixel patches from 4167 calibrated 1020 × 1532 images. Each 3 × 3 pixel patch obviously a vector in $\mathbb{R}^9$. Normalized to constant intensity and to unit euclidean norm. Transformed by a basis choice that highlights geometric features of the dataset itself. Result lies on the unit 7-sphere in $\mathbb{R}^8$. 

Mumford dataset

We use the smoothing procedure developed by Jennifer Kloke. Once smoothed to a circle, we parametrize with persistent cohomology, and can pull the parametrization back to the original data points.
Mumford dataset
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