Persistent cohomology and period reconstruction

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Outline

Persistent cohomology and circular coordinates

Periodic systems and period reconstruction
Circular coordinates

From algebraic topology, we know the isomorphism, natural in $X$

$$H^1(X; \mathbb{Z}) \cong [X, S^1]$$

We can use this and a smoothing process to go from computed coclasses for a point cloud to circle-valued coordinate maps.

Reference

Examples
Examples
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Periodic systems and period reconstruction
Periodic systems form circles

Completely described periodic (or even recurrent) systems form circles embedded into phase space.

Example

An oscillating system swings according to the system

\[
\frac{d^2 \theta}{dt^2} + a\theta = 0
\]

and thus has position and velocity described by

\[
\theta(t) = \frac{1}{a} \sin(\omega t) \quad \theta'(t) = -\cos(\omega t)
\]

describing an ellipse in the $\theta-\theta'$-plane.
Coordinatization describes recurrence

Coordinatizing a periodic (or recurrent) system yields a description of the periodicity (recurrence).

Example (cont. . .)

A circular coordinate function, recovered by sampling and a persistent cohomology computation would be

\[(\theta(t), \theta'(t)) \mapsto \frac{a}{2\pi} t \pmod{1.0}\]

The coordinate space we use is \(S^1 = [0, 1]/(0 \sim 1)\).
Unrolling the coordinates
Unrolling the coordinates
Unrolling the coordinates
Unrolling the coordinates
Unrolling the coordinates

\[ y = 0.5 \times x \]

Correlation: 1.0
Recovering period lengths from delay embeddings

Dataset


Approach

Embed $t \mapsto (a_t, a_{t+\epsilon}, a_{t+2\epsilon}) \in \mathbb{R}^3$.

Yields curve from timeseries.

If timeseries is periodic, so is the resulting embedding.
Anderson air temperature data
Anderson air temperature data
Anderson air temperature data

- Persistent cohomology
- Circular coordinates
- Periodic systems
- Period reconstruction

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Anderson air temperature data

\[ x = -12.00y + 2.56 \]

Correlation: -0.9998
Anderson air temperature data

Average air temperature @ Nottingham Castle

Month in a given period

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Persistent cohomology and circular coordinates
Periodic systems and period reconstruction
Why topological methods?

There already are methods used in dynamical systems and in signals processing for determining periodicity and periods. We believe this system to be

- Less noise-sensitive: Fourier analysis requires actually periodic data and will easily miss periods with noise.
- Less noise-introducing: periods may be recognized by simply taking differences of points at different estimated period lengths. This is well-known to introduce extra noise.
- Multidimensional by design: phase spaces of arbitrary high dimensions are easy to deal with.
Functoriality chooses delays

Changing delay from $\epsilon_0$ to $\epsilon_1$ yields a map of point clouds:

$$(x_t, x_{t+\epsilon_0}, x_{t+2\epsilon_0}) \mapsto (x_t, x_{t+\epsilon_1}, x_{t+2\epsilon_1})$$

By functoriality: map in homology.

Thus we can consider persistence of features of delay embeddings over different delays; and use this to pick good delay values.

Periodic processes look like circles. Thus we look for a single dominant $\beta_1$ interval.