# Computation of spectral sequences of double complexes with applications to persistent homology

Mikael Vejdemo-Johansson
joint with David Lipsky, Dmitriy Morozov, Primoz Skraba

School of Computer Science
University of St Andrews
Scotland

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# Outline

# Double complex

Fix some nice commutative ring $R$. Everything in this talk will take place in $R - \mathrm{Mod}$; maybe $R - \mathrm{grMod}$.

## Definition

Let $C_{*,*}$ be a bigraded module over $R$. $C_{*,*}$ is a double complex if there are maps $d' : C_{*,*} \to C_{*-1,*}$ and $d'' : C_{*,*} \to C_{*,*-1}$ such that $d'd' = 0$, $d''d'' = 0$ and $d'd'' + d''d' = 0$.

# Total complex

## Definition

If $C_{*,*}$ is a double complex, $\mathrm{Tot}(C)$ is a chain complex

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

with differential $d = d' + d''$.

# Spectral sequence

For every double complex $C_{*,*}$, there are two spectral sequences with $E^0 = C_{*,*}$.

**Horizontal spectral sequence**

$d^0 = d'$. $d^1$ induced from $d''$.

**Vertical spectral sequence**

$d^0 = d''$. $d^1$ induced from $d'$.

**Homology of the total complex (Godement, 1958)**

For each of these spectral sequences, the $E^\infty$-page is the graded associated module of a filtration on $H_*(\mathrm{Tot}(C))$.

# How do we compute this?

## Bott and Tu

Gives a few fundamental results; in a de Rham cohomology context.

## Godement

Gives some theorems, does not give higher differentials.

## McCleary

Gives some theorems, no explicit higher differentials.

## Brown

Has double complex s.s. as an example. Higher differentials are "hard".

## MacLane

Gives some theorems on the double complex, no explicit higher differentials.

# Outline

# Independent definition of cycles and boundaries

## $Z^r_{p,q}$

Consists of $\alpha_p$ such that there are $\alpha_{p-1}, \dots, \alpha_{p-r}$ fulfilling equations

$$d^0\alpha_p = 0 \qquad d^0\alpha_{p-1} = d^1\alpha_p \qquad \dots \qquad d^0\alpha_{p-r} = d^1\alpha_{p-r+1}$$

## $B^r_{p,q}$

Consists of $\alpha_p$ such that there are $\beta_p, \beta_{p+1}, \dots, \beta_{p+r}$ fulfilling equations

$$d^0\beta_p + d^1\beta_{p+1} = \alpha_p \quad d^0\beta_{p+2} = d^1\beta_{p+1} \quad \dots \quad d^0\beta_{p+r-1} = d^1\beta_{p+r}$$

Partial collection of conditions on $(\alpha_0, \alpha_1, \dots, \alpha_p, 0, \dots, 0)$ being a cycle in $\mathrm{Tot}(C)$; and on it being a boundary.

# Fundamental computational steps

The higher differentials are decomposed into smaller computational steps:

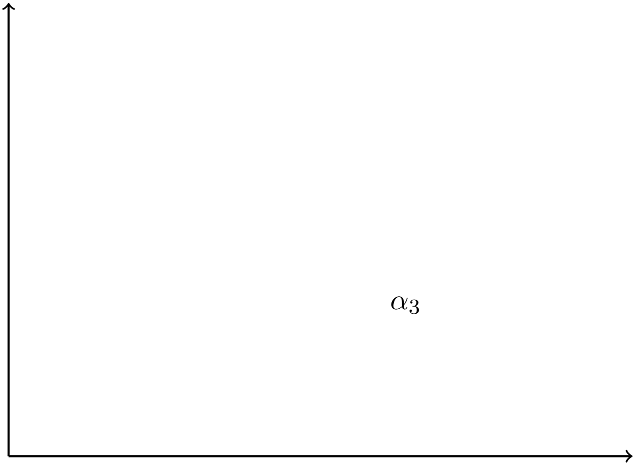
### Across

$v \in Z^r_{p,q}$ maps by a map induced from $d^1$ to $d^1 v \in Z^r_{p-1,q}$.

### Up

$d^0$ has sections over $B^1_{p,q}$. Hence, for $r \geq 1$, $w \in B^r_{p,q}$ maps by a map induced from $d^0$ to $(d^0)^{-1} w \in Z^r_{p,q+1}$.

# The $E^r$ differential



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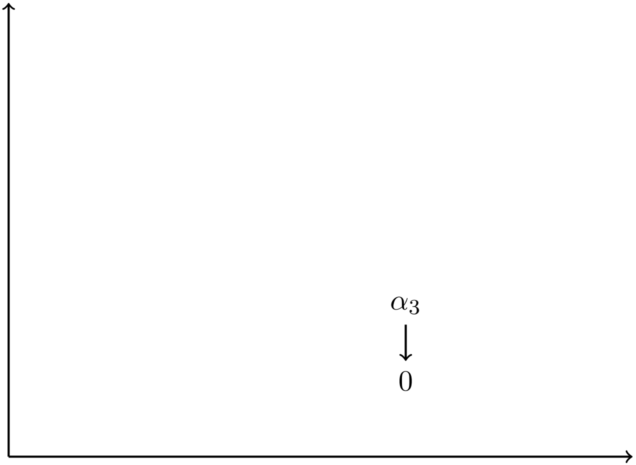

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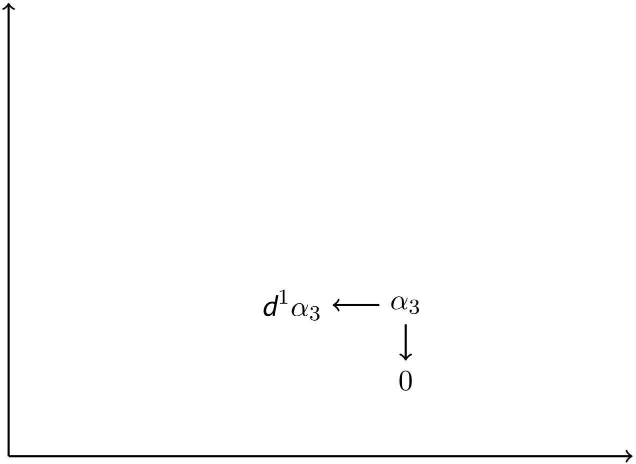

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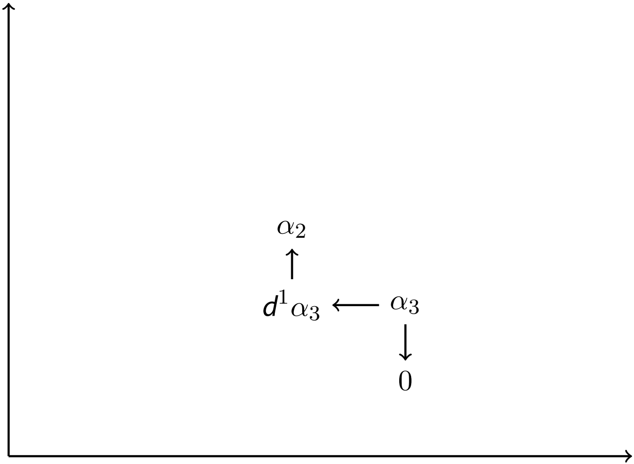

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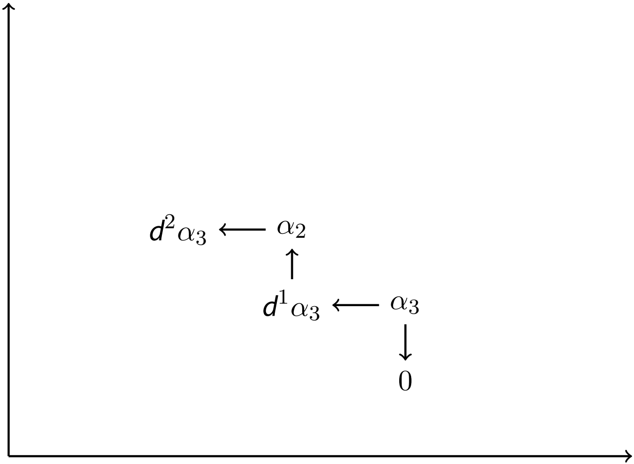

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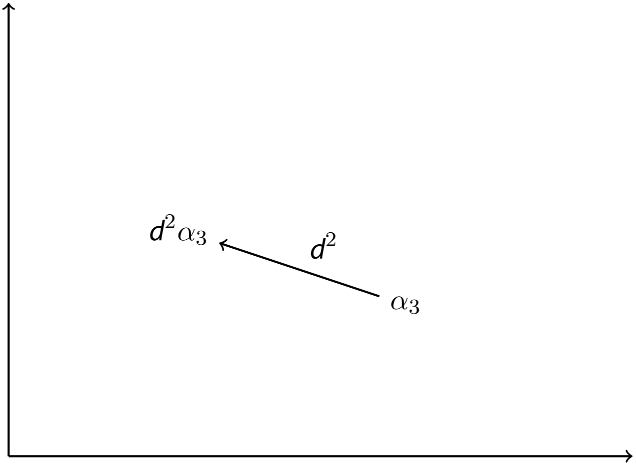

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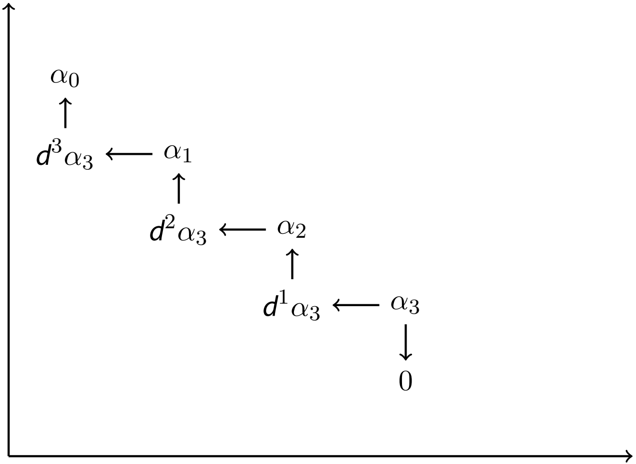
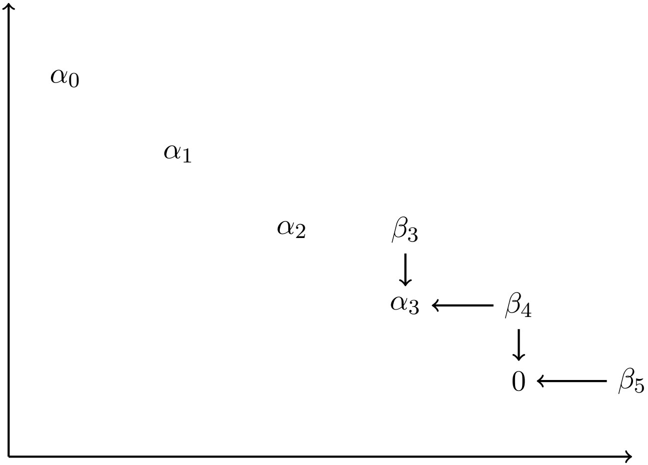


# The $E^r$ differential



# Outline

# Primer on persistence language

persistence module
: graded module over $\Bbbk[t]$

barcode
: decomposition of a persistence module into free and torsion summands

Smith normal form is unreasonably effective for $\Bbbk[t]$ – no GCD computations needed; lowest degree immediately implies divisibility.

# Representative chains in $\mathrm{Tot}(C)$

$E^\infty$ has a basis on the shape

$$\phi(\alpha_p) = (\alpha_0, \dots, \alpha_{p-1}, \alpha_p, 0, \dots, 0)$$

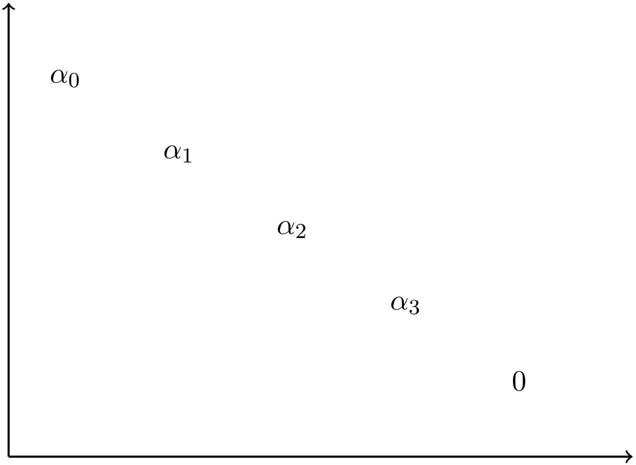
Write $A_p$ for such $\alpha$ in $E^\infty_{p,q}$.

**Problem**

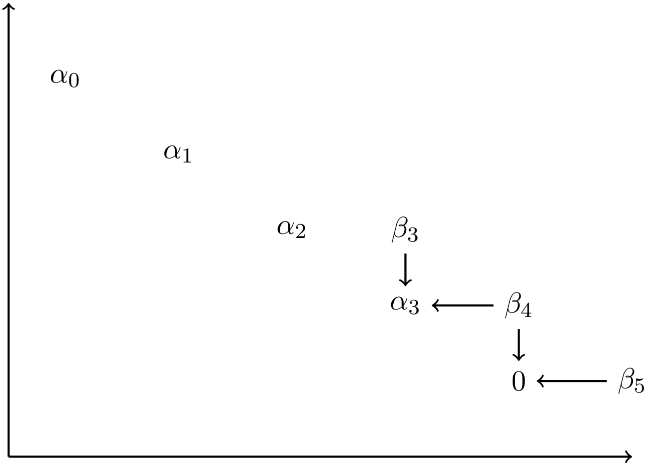
$t^s\alpha_p \in B^\infty_{p,q}$ does not necessarily mean $\phi(t^s\alpha_p) = 0$.

So behavior of barcodes for $\alpha_p$ is different from behavior of barcodes for $\phi(\alpha_p)$.

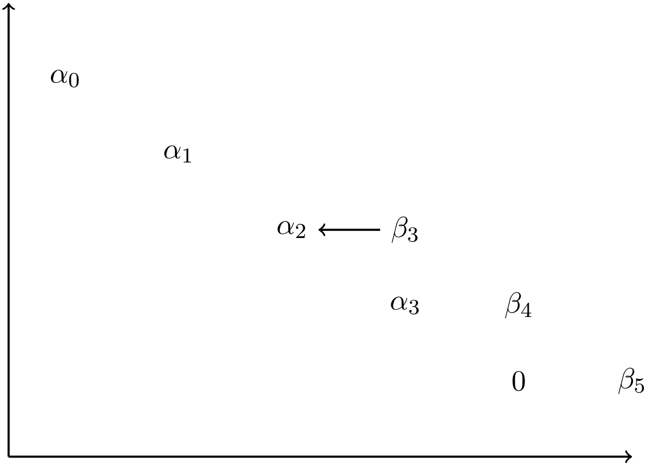
# Finding coefficients



# Finding coefficients



# Finding coefficients



# Finding coefficients

Consider

$$\phi(\alpha_p) = (\phi_0(\alpha_p), \ldots, \phi_{p-1}(\alpha_p), \alpha_p, 0, \ldots, 0)$$

Suppose

$$t^s \alpha_p \in B^\infty$$

There is some element

$$\beta = (0, \ldots, \beta_p, \ldots, \beta_n)$$

Such that

$$\phi(t^s \alpha_p) - D\beta = (t^s \phi_0(\alpha), \ldots, t^s \phi_{p-1}(\alpha) - d^1 \beta_p, 0, \ldots, 0)$$

# Finding coefficients

Express this result with our preferred basis:

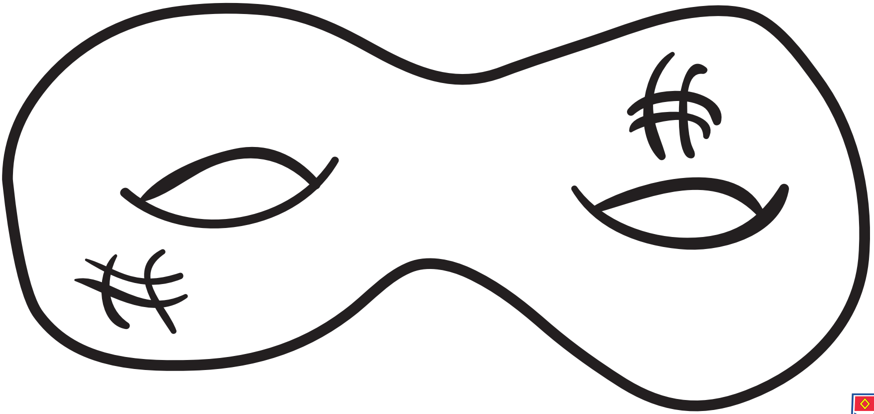
$$\begin{aligned} t^s\phi(\alpha) - D\beta &= (t^s\phi_0(\alpha), \dots, t^s\phi_{p-1}(\alpha) - d^1\beta_p, 0, \dots, 0) \\ &= \sum_{\alpha' \in A_{n-1}} c'\phi(\alpha') + \sum_{\alpha'' \in A_{n-2}} c''\phi(\alpha'') + \cdots + \sum_{\alpha^{(n)} \in A_0} c^{(n)}\phi(\alpha^{(n)}) \end{aligned}$$

Collect all these $c', c'', \dots, c^{(n)}$ into a matrix. This matrix gives relations for $H_*(\mathrm{Tot}(C))$.

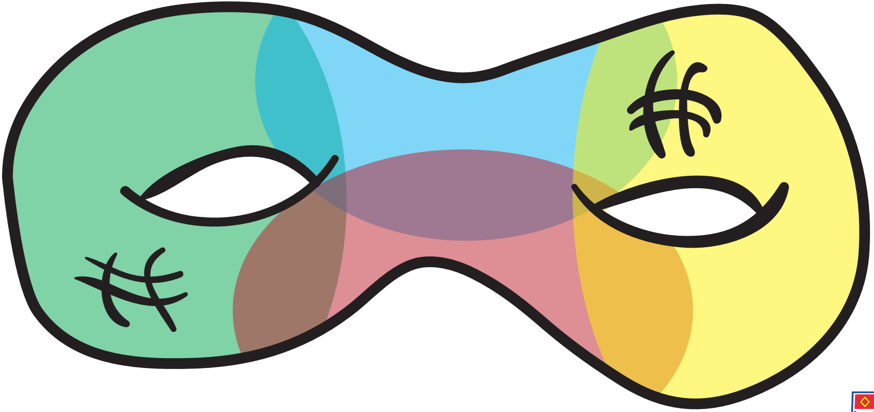
Smith normal form on this presentation matrix gives global barcodes.

# Outline

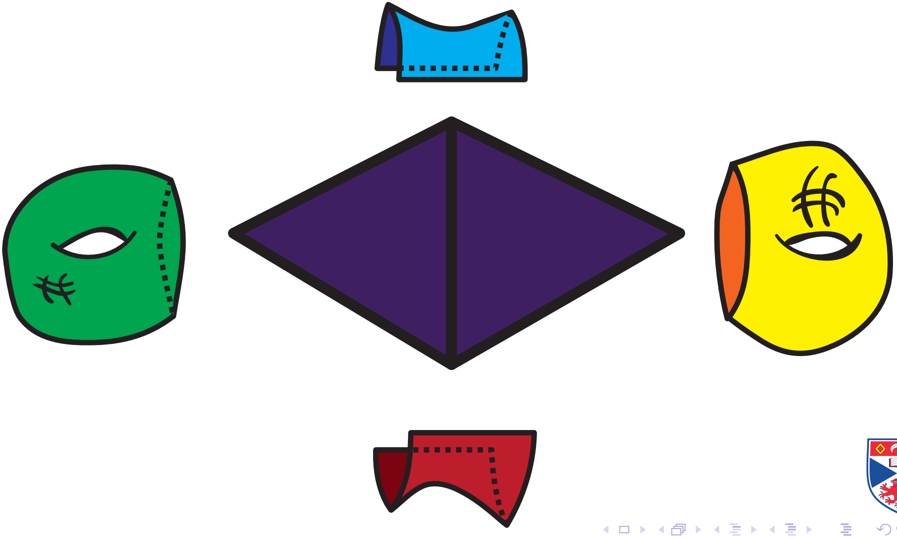
# Mayer-Vietoris spectral sequence



# Mayer-Vietoris spectral sequence



# Mayer-Vietoris spectral sequence



# Čech spectral sequence

**Double complex**

$i$th column
: Chain complex of all $i$-fold intersections

vertical differential
: Boundary in chain complex

horizontal differential
: Inclusion of deeper intersection into more shallow; coefficient from Čech complex boundary.

Generalizes the Mayer-Vietoris long exact sequence.

# Čech spectral sequence

### Theorem

This double complex computes $H_*X$ using only a cover of $X$.

### Proof sketch (following Brown; also Segal)

$$\begin{array}{ccccc}
C_2 \bigsqcup_\sigma \bigcap_{j=0}^0 U_{\sigma_j} & \leftarrow & C_2 \bigsqcup_\sigma \bigcap_{j=0}^1 U_{\sigma_j} & \leftarrow & C_2 \bigsqcup_\sigma \bigcap_{j=0}^2 U_{\sigma_j} \\
\downarrow & & \downarrow & & \downarrow \\
C_1 \bigsqcup_\sigma \bigcap_{j=0}^0 U_{\sigma_j} & \leftarrow & C_1 \bigsqcup_\sigma \bigcap_{j=0}^1 U_{\sigma_j} & \leftarrow & C_1 \bigsqcup_\sigma \bigcap_{j=0}^2 U_{\sigma_j} \\
\downarrow & & \downarrow & & \downarrow \\
C_0 \bigsqcup_\sigma \bigcap_{j=0}^0 U_{\sigma_j} & \leftarrow & C_0 \bigsqcup_\sigma \bigcap_{j=0}^1 U_{\sigma_j} & \leftarrow & C_0 \bigsqcup_\sigma \bigcap_{j=0}^2 U_{\sigma_j}
\end{array}$$

# Čech spectral sequence

## Theorem

This double complex computes $H_*X$ using only a cover of $X$.

## Proof sketch (following Brown; also Segal)

$$
\begin{array}{ccccccc}
C_2X & \longleftarrow & C_2 \coprod_\sigma \bigcap_{j=0}^0 U_{\sigma_j} & \leftarrow & C_2 \coprod_\sigma \bigcap_{j=0}^1 U_{\sigma_j} & \leftarrow & C_2 \coprod_\sigma \bigcap_{j=0}^2 U_{\sigma_j} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1X & \longleftarrow & C_1 \coprod_\sigma \bigcap_{j=0}^0 U_{\sigma_j} & \leftarrow & C_1 \coprod_\sigma \bigcap_{j=0}^1 U_{\sigma_j} & \leftarrow & C_1 \coprod_\sigma \bigcap_{j=0}^2 U_{\sigma_j} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_0X & \longleftarrow & C_0 \coprod_\sigma \bigcap_{j=0}^0 U_{\sigma_j} & \leftarrow & C_0 \coprod_\sigma \bigcap_{j=0}^1 U_{\sigma_j} & \leftarrow & C_0 \coprod_\sigma \bigcap_{j=0}^2 U_{\sigma_j}
\end{array}
$$

Augment double complex

# Čech spectral sequence

## Theorem

This double complex computes $H_*X$ using only a cover of $X$.

## Proof sketch (following Brown; also Segal)

$$C_2X \longleftarrow C_2 \coprod_\sigma \bigcap_{j=0}^0 U_{\sigma_j} \leftarrow C_2 \coprod_\sigma \bigcap_{j=0}^1 U_{\sigma_j} \leftarrow C_2 \coprod_\sigma \bigcap_{j=0}^2 U_{\sigma_j}$$

$$C_1X \longleftarrow C_1 \coprod_\sigma \bigcap_{j=0}^0 U_{\sigma_j} \leftarrow C_1 \coprod_\sigma \bigcap_{j=0}^1 U_{\sigma_j} \leftarrow C_1 \coprod_\sigma \bigcap_{j=0}^2 U_{\sigma_j}$$

$$C_0X \longleftarrow C_0 \coprod_\sigma \bigcap_{j=0}^0 U_{\sigma_j} \leftarrow C_0 \coprod_\sigma \bigcap_{j=0}^1 U_{\sigma_j} \leftarrow C_0 \coprod_\sigma \bigcap_{j=0}^2 U_{\sigma_j}$$

Resolution (in chain complexes) – because any simplex $\sigma \in C_*X$ is $\sigma \times \Delta^k$ in the double complex. Thus: associated graded $H_*\mathrm{Tot} = 0$.

# Čech spectral sequence

## Theorem

This double complex computes $H_*X$ using only a cover of $X$.

## Proof sketch (following Brown; also Segal)

$$\begin{array}{cccc}
C_2X & C_2\coprod_\sigma \bigcap_{j=0}^{0} U_{\sigma_j} & C_2\coprod_\sigma \bigcap_{j=0}^{1} U_{\sigma_j} & C_2\coprod_\sigma \bigcap_{j=0}^{2} U_{\sigma_j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
C_1X & C_1\coprod_\sigma \bigcap_{j=0}^{0} U_{\sigma_j} & C_1\coprod_\sigma \bigcap_{j=0}^{1} U_{\sigma_j} & C_1\coprod_\sigma \bigcap_{j=0}^{2} U_{\sigma_j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
C_0X & C_0\coprod_\sigma \bigcap_{j=0}^{0} U_{\sigma_j} & C_0\coprod_\sigma \bigcap_{j=0}^{1} U_{\sigma_j} & C_0\coprod_\sigma \bigcap_{j=0}^{2} U_{\sigma_j}
\end{array}$$

Associated graded $H_*\mathrm{Tot} = 0$ does not change with a change in filtration.

# Čech spectral sequence

## Theorem

This double complex computes $H_*X$ using only a cover of $X$.

## Proof sketch (following Brown; also Segal)

$$\begin{array}{ccccccc}
C_2X & & C_2 \coprod_\sigma \bigcap_{j=0}^{0} U_{\sigma_j} & \leftarrow & C_2 \coprod_\sigma \bigcap_{j=0}^{1} U_{\sigma_j} & \leftarrow & C_2 \coprod_\sigma \bigcap_{j=0}^{2} U_{\sigma_j} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1X & & C_1 \coprod_\sigma \bigcap_{j=0}^{0} U_{\sigma_j} & \leftarrow & C_1 \coprod_\sigma \bigcap_{j=0}^{1} U_{\sigma_j} & \leftarrow & C_1 \coprod_\sigma \bigcap_{j=0}^{2} U_{\sigma_j} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_0X & & C_0 \coprod_\sigma \bigcap_{j=0}^{0} U_{\sigma_j} & \leftarrow & C_0 \coprod_\sigma \bigcap_{j=0}^{1} U_{\sigma_j} & \leftarrow & C_0 \coprod_\sigma \bigcap_{j=0}^{2} U_{\sigma_j}
\end{array}$$

Because resolution: removing augmentation yields quasi-isomorphism.

# Parallel/distributed computation of persistent homology

**Strategy**

**Divide** a filtered simplicial complex $X$ into a cover of filtered simplicial subcomplexes $U_i$.

**Conquer** by merging all $H_*(\bigcap_i U_i)$ using the Mayer-Vietoris spectral sequence.

# Parallel/distributed computation of persistent homology

## Problems

We need to build tools for efficient computation.

## Hope

Algorithm asymptotically equally fast as persistent homology; but with memory consumption guarantees.

## Results

Achieved this for the two-column case – deepest intersection pairwise.

Note: persistence can run in $O(nm^2)$ time; for $n$ simplices, $m$ cycles. We can now achieve $O(n^2m)$ for the two-column case.

Matrix fill-in and distributed module algebra our hardest problems.

# More spectral sequences persistently

## Results summary

- Explicit higher differentials for double complex spectral sequences
- Explicit extensions for persistence modules

## Serre spectral sequences

Dress (1967) constructs the Serre spectral sequence as a double complex spectral sequence.

- All higher differentials follow
- Applicability in persistence follows
- Open question: how do we make this finite enough?

# Thank you

## Happy birthdays!

to Gunnar Carlsson, Ralph Cohen, and Ib Madsen



## Acknowledgements

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Questions?