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# Towards a topos foundation for persistent homology

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SEVENTH FRAMEWORK PROGRAMME

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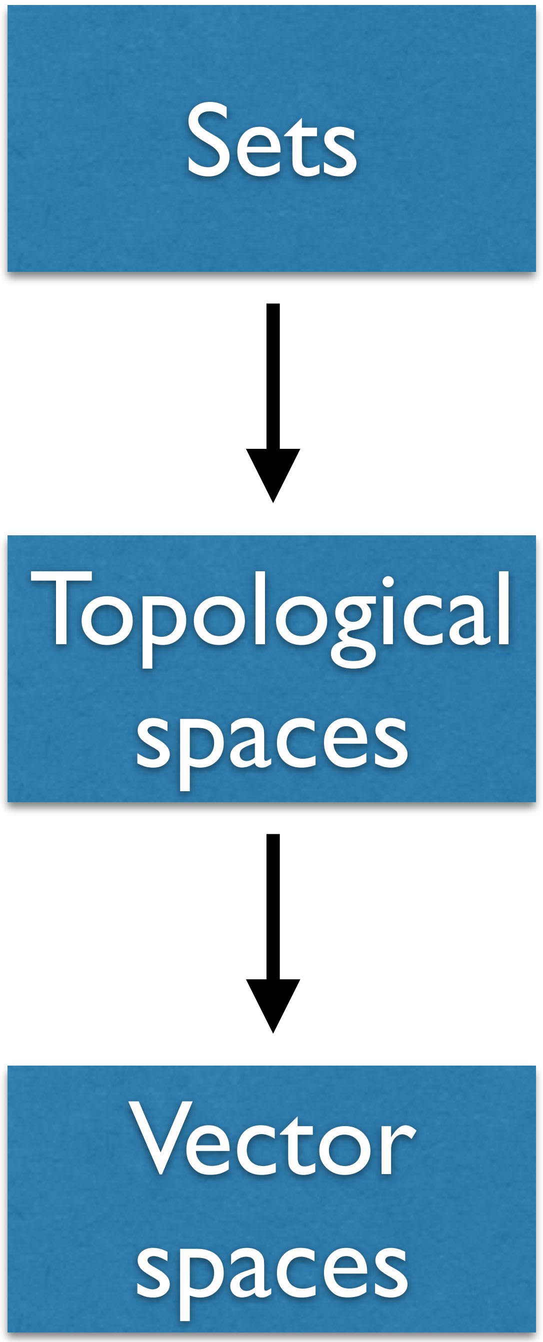
# Current algebraic approaches

- Most algebraic approaches appear late in the "stack" of abstractions:
- Diagrams of vector spaces
- Diagrams of topological spaces



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# Current algebraic approaches

- Most algebraic approaches appear late in the "stack" of abstractions:
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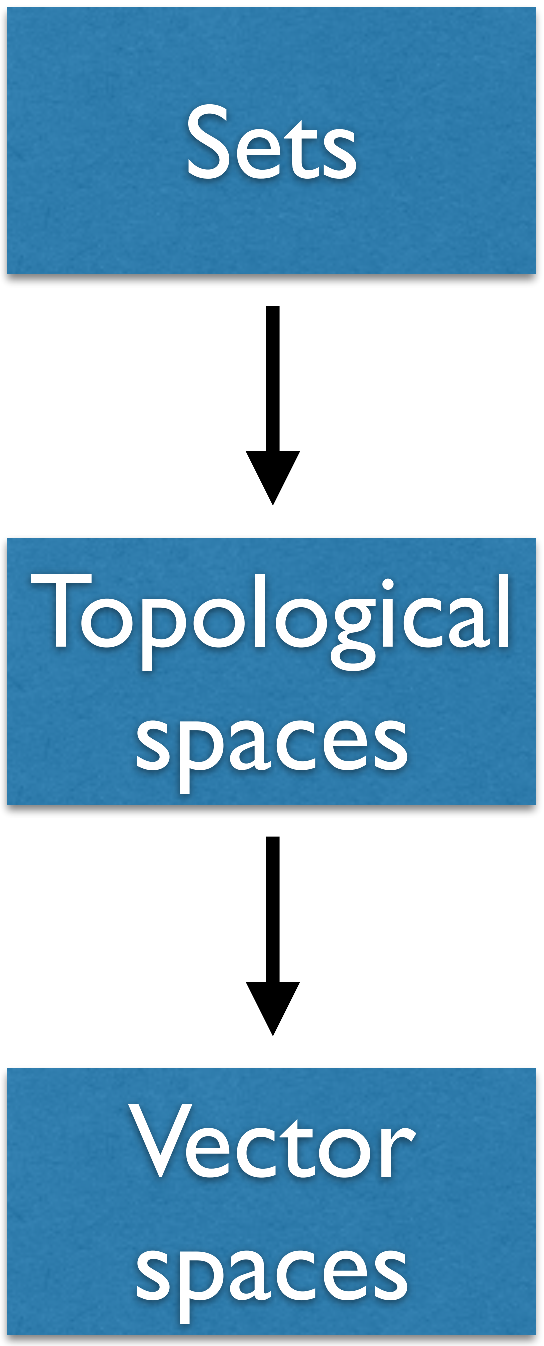
- Our idea: introduce persistence to the stack at the level of Set Theory



# How do we change sets?

- Topos:
  A category with enough structure to work like the category Set.
- Proposition:
  The category of Set-valued sheaves over a site form a topos.
- Observation [Barr&Wells]:
  By picking a site, we choose a *shape* for our sets.
  We can use this for time-variant or fuzzy sets.

# Sheaves and cosheaves

More and more researchers are using sheaves or cosheaves to represent persistence modules and persistent (co)homology

- J Curry:
  Sheaves, cosheaves and applications
- V de Silva, E Munch, A Patel:
  Categorified reeb graphs
- R Ghrist, M Robinson, S Krishnan, V Nanda, …

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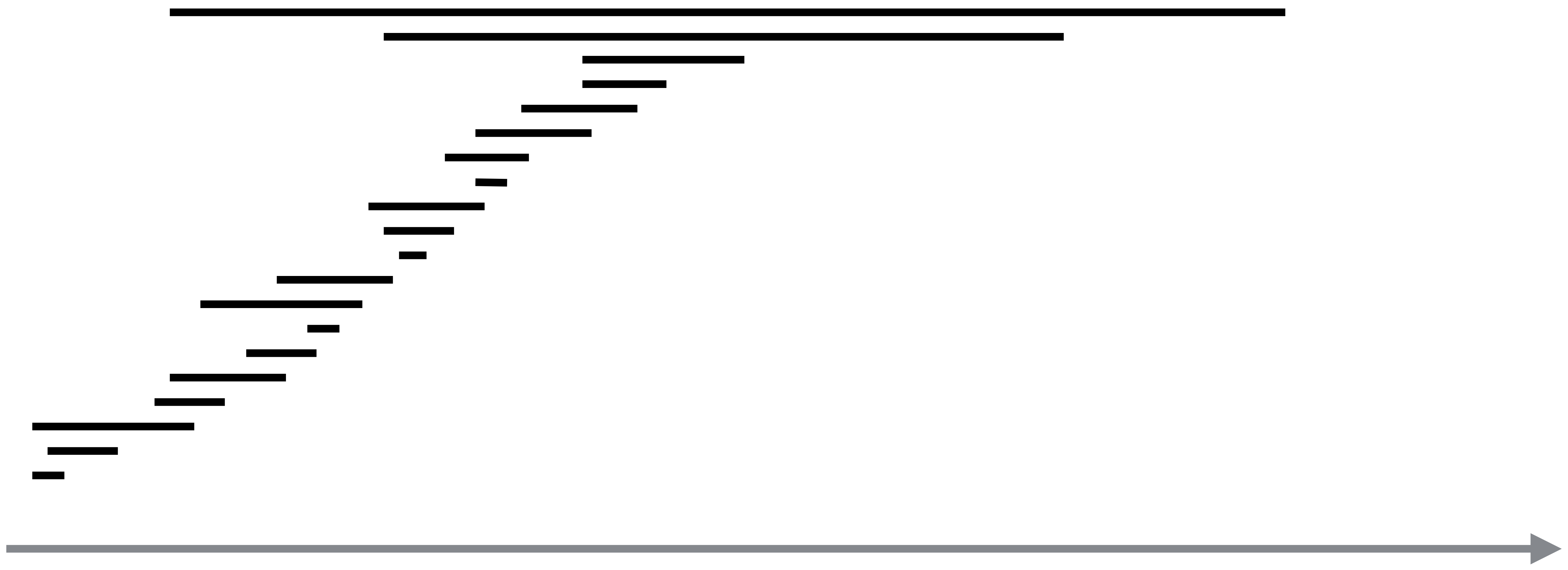
# Why is persistence about sheaves?



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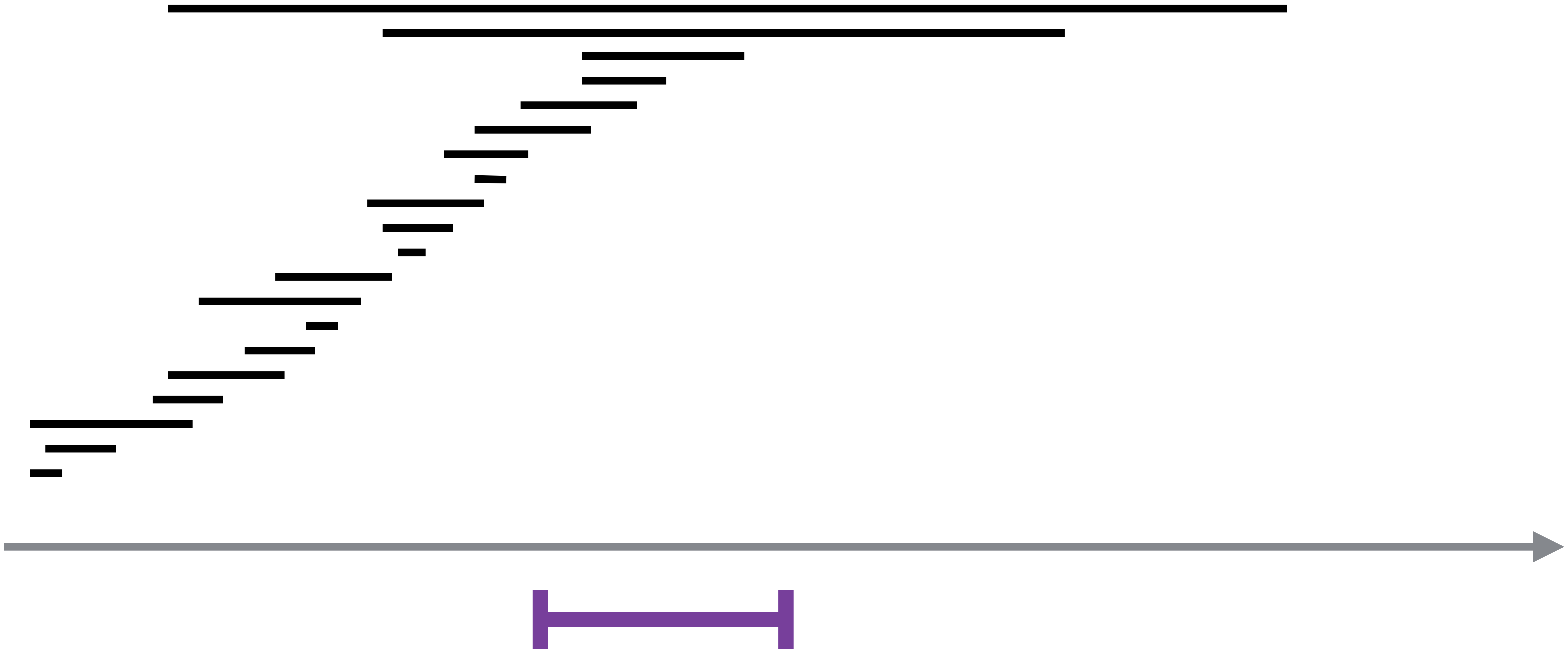

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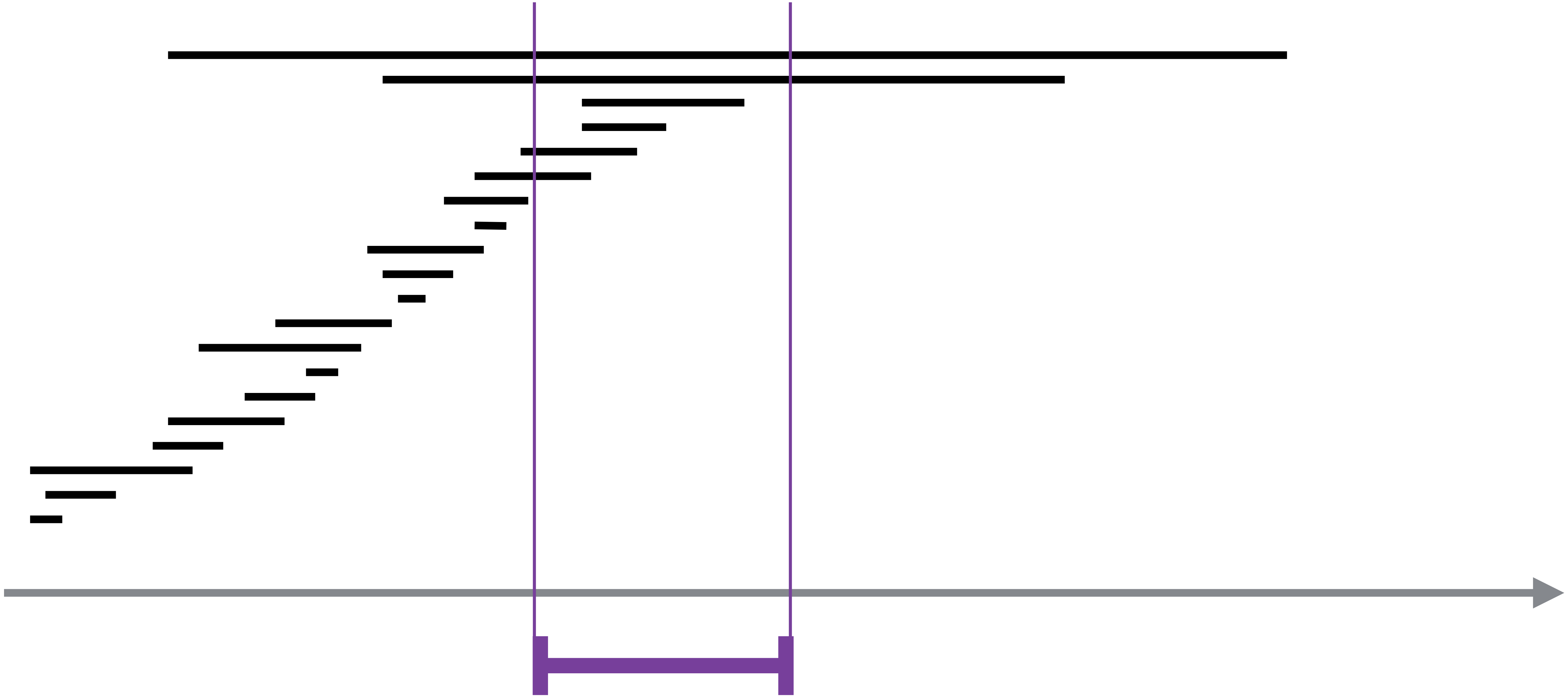

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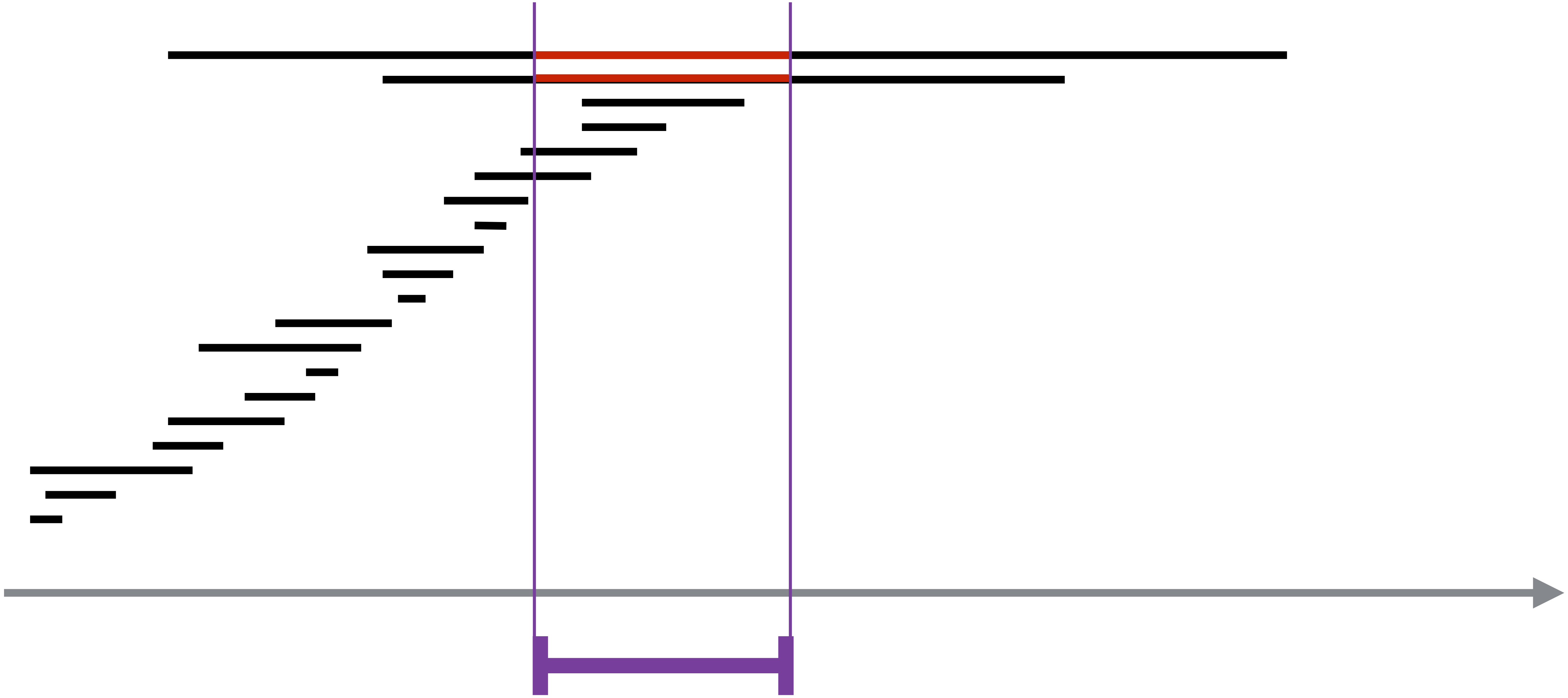

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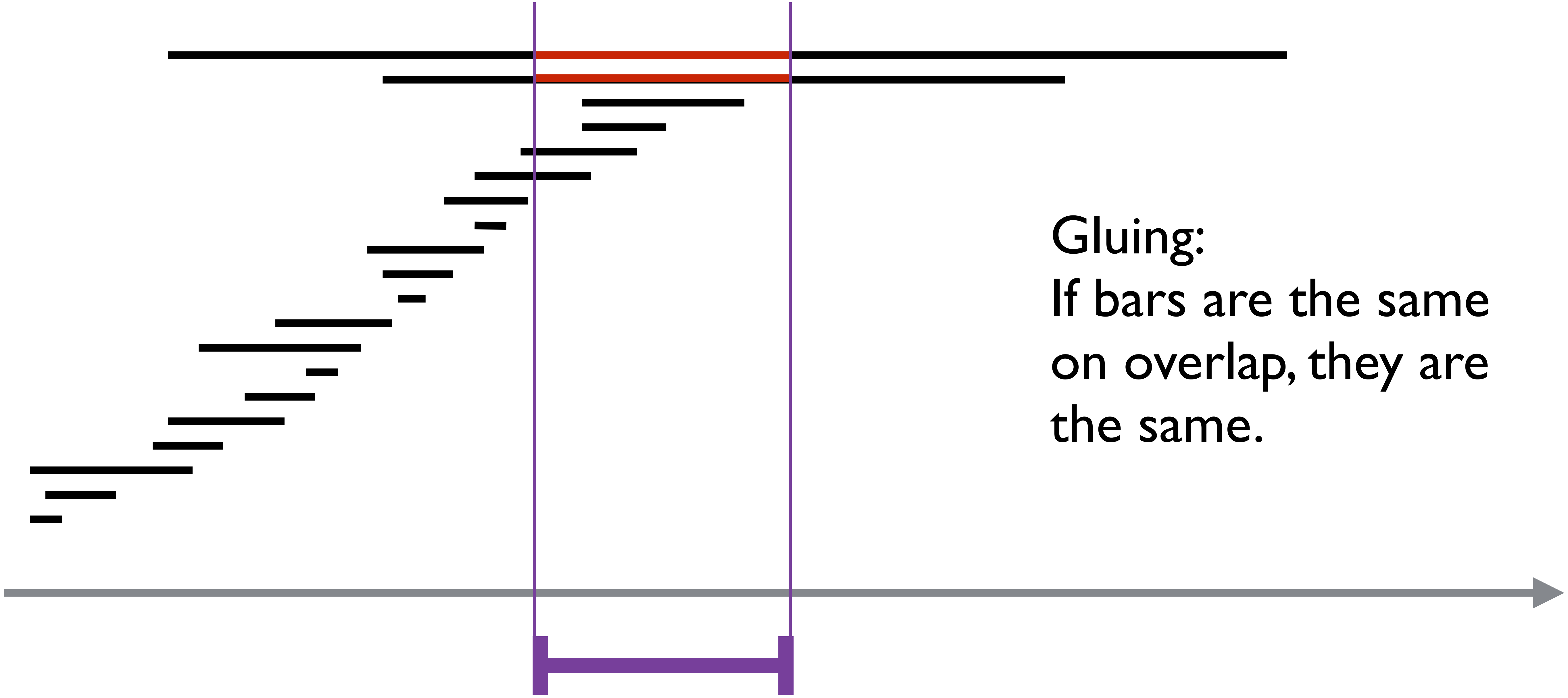

Gluing:
If bars are the same on overlap, they are the same.

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# Sheaves over *what*?

Need a site to build a sheaf

- Curry: Cell decomposition of $\mathbb{R}$
- de Silva-Munch-Patel: $\mathbb{R}$

Problem:
$\mathbb{R}$ has disconnected open sets.
We could have persistent objects that flicker in and out of existence

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# Sheaves over *what*?

Maybe we should change our site?

Features we want:

- Intervals as basic “query” objects
- No “flickering”: union of disjoint intervals should be covering interval
- Intersections as we are used to

# Sheaves over *what*?

We still need for it to be a site.
Our specification is not a topological space — unions of opens are not necessarily open.

Sites for topoi of sheaves:

Topological spaces are not minimal requirements — we can use *Heyting Algebras* instead.

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# Heyting algebra

- A Heyting algebra is a complete distributive lattice: Set with operations ∧ and ∨, elements ⊤ and ⊥ subject to rules.
- It is a partial order through
  x ≤ y ⇔ x∧y = x
- Rules include the infinite distributive law

$$x \wedge \bigvee_{k=1}^{\infty} y_k = \bigvee_{k=1}^{\infty} (x \wedge y_k)$$

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# Heyting algebra vs. Topological Spaces

| **Heyting Algebra** | **Topology on X** |
|---|---|
| $U \wedge V$ | $U \cap V$ |
| $U \vee V$ | $U \cup V$ |
| $\bot$ | $\varnothing$ |
| $\top$ | $X$ |

# Operations overview

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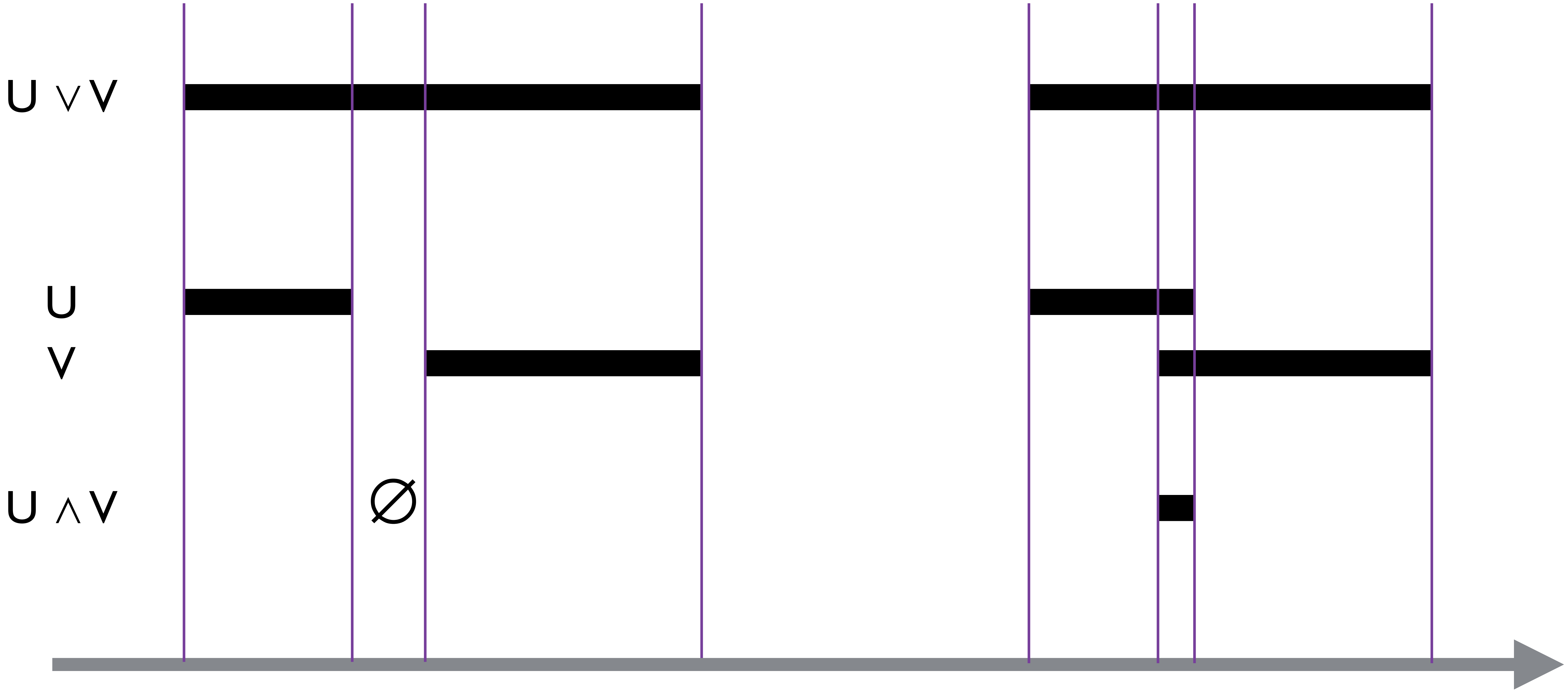

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# Not distributive!

Turns out that these operations definitions do not make a distributive lattice.

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# Heyting Algebra of Persistence Lifetimes

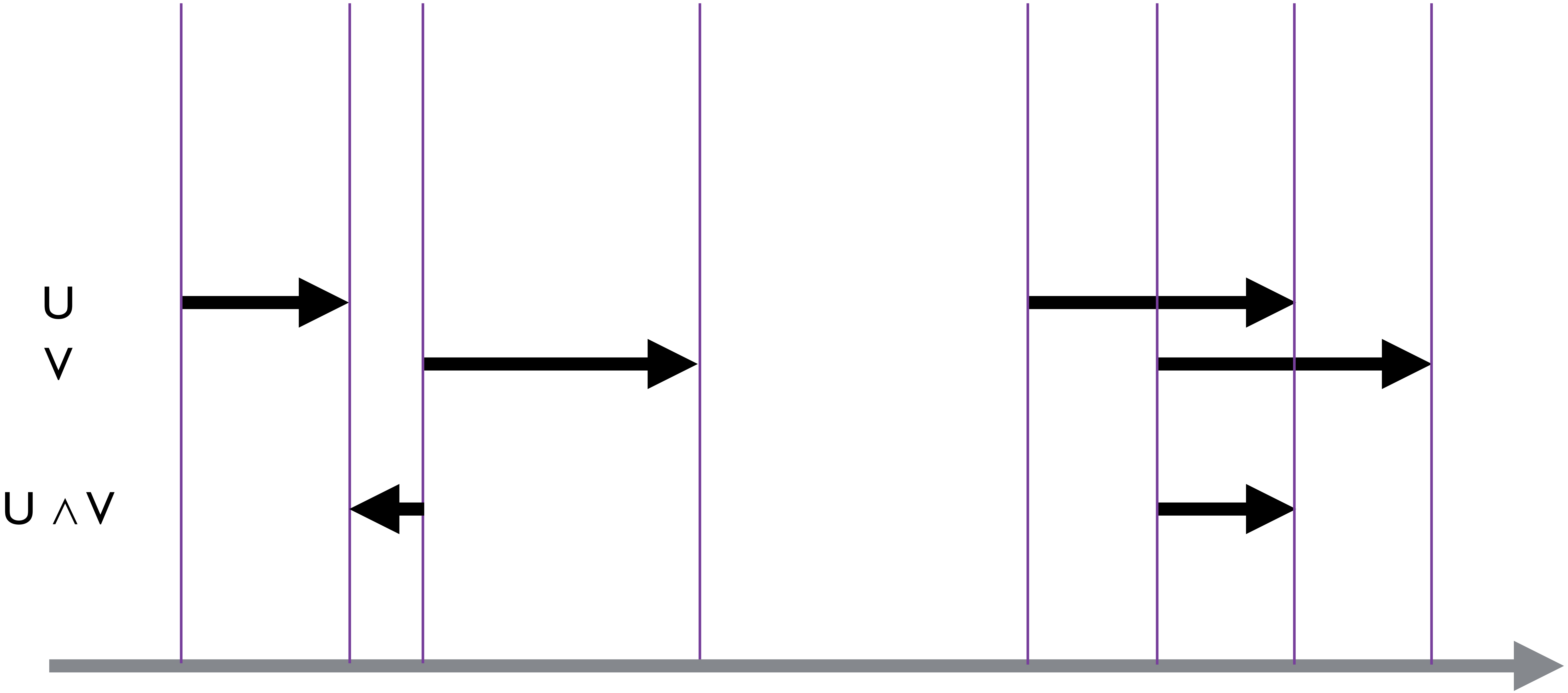
Instead, we find a Heyting Algebra structure P by considering the collection of *directed intervals* in $\mathbb{R}$.

Intervals go forwards or backwards.

As a partial order, we use $\mathbb{R} \times \mathbb{R}^{op}$.
To get $\top$ and $\bot$ to work out, we include $(\pm\infty, \pm\infty)$.

We call the resulting topos **PSet** = $Sh_{Set}(P)$.

# Revised operations



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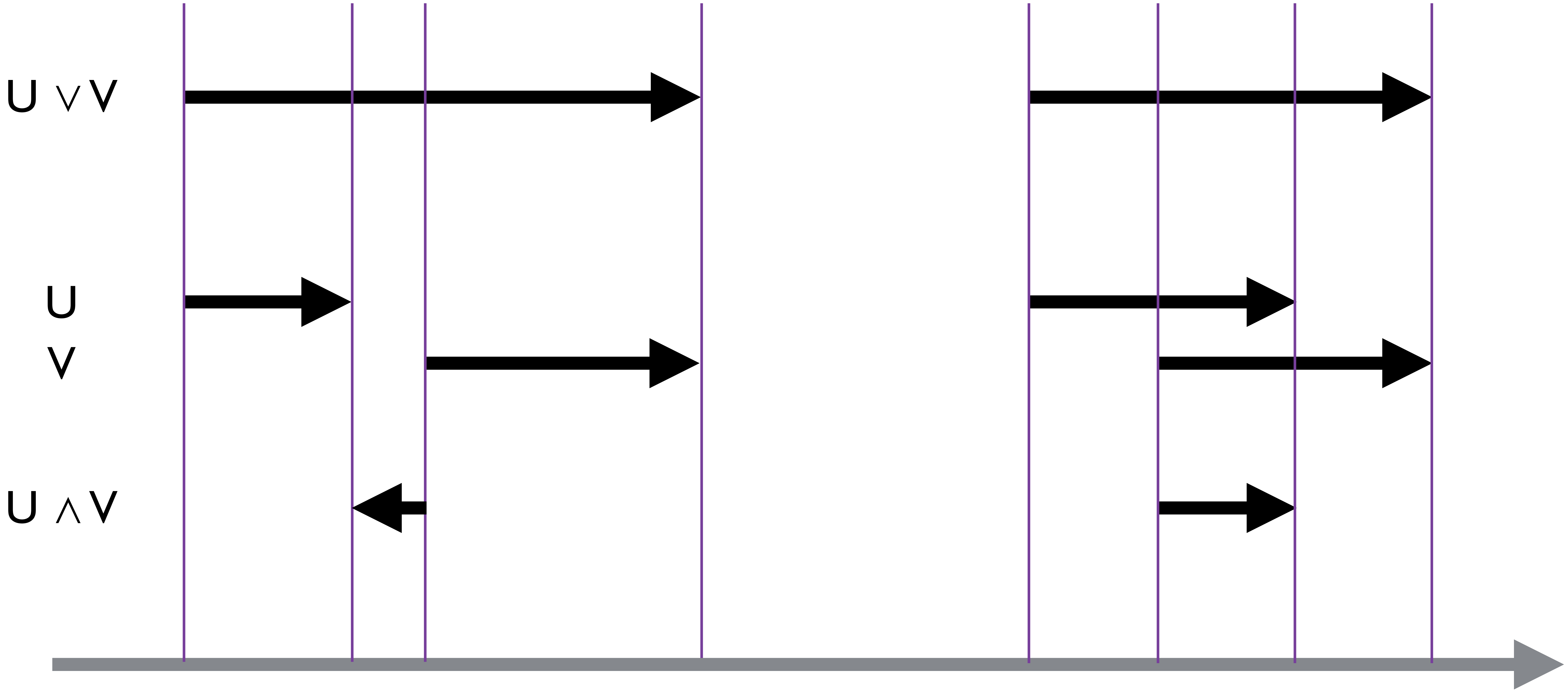
# Revised operations



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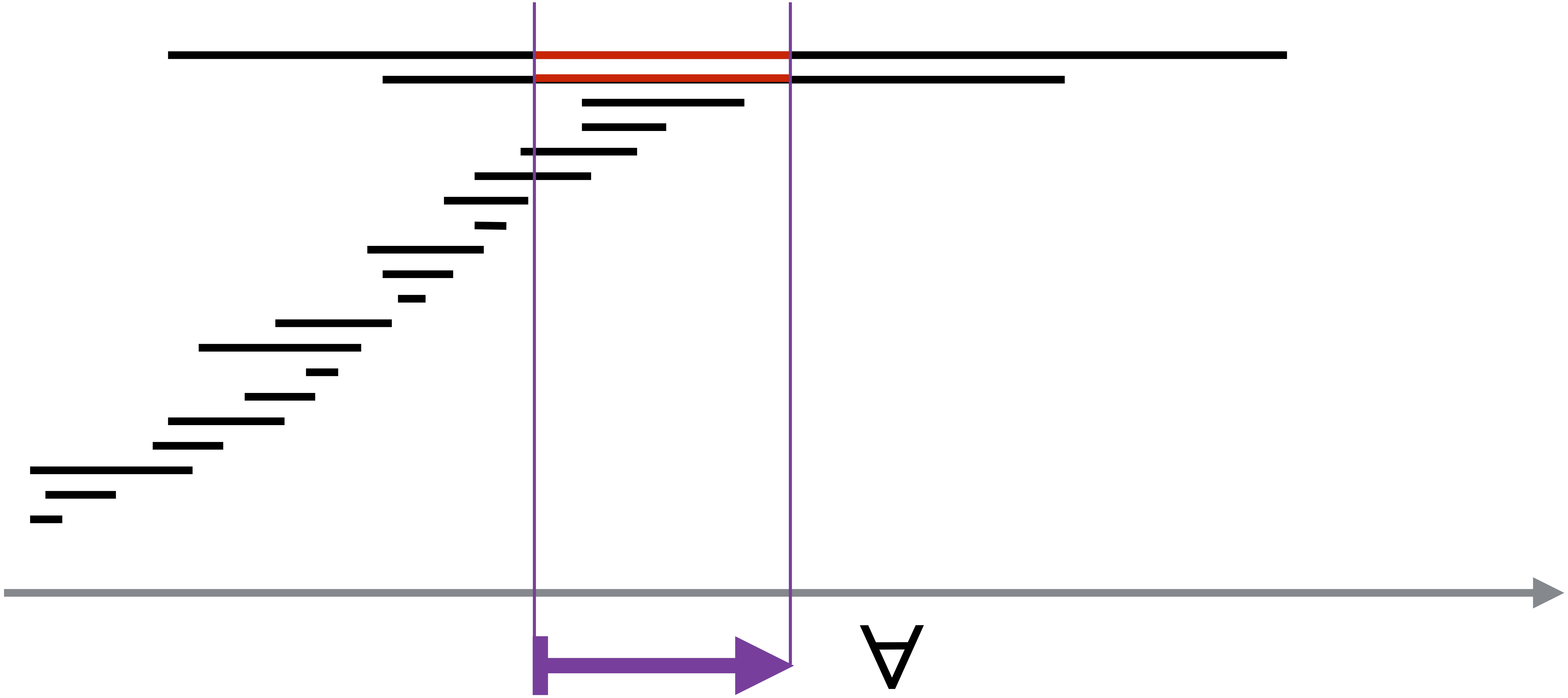
# Semantics of the arrow directions

Forward intervals should still be interpreted as representing **Things that exist during the entire interval.**

Backward intervals can be interpreted as representing **Things that exist at some point in the interval.**

These semantics work with all the resulting inclusion maps.

# Elements over intervals



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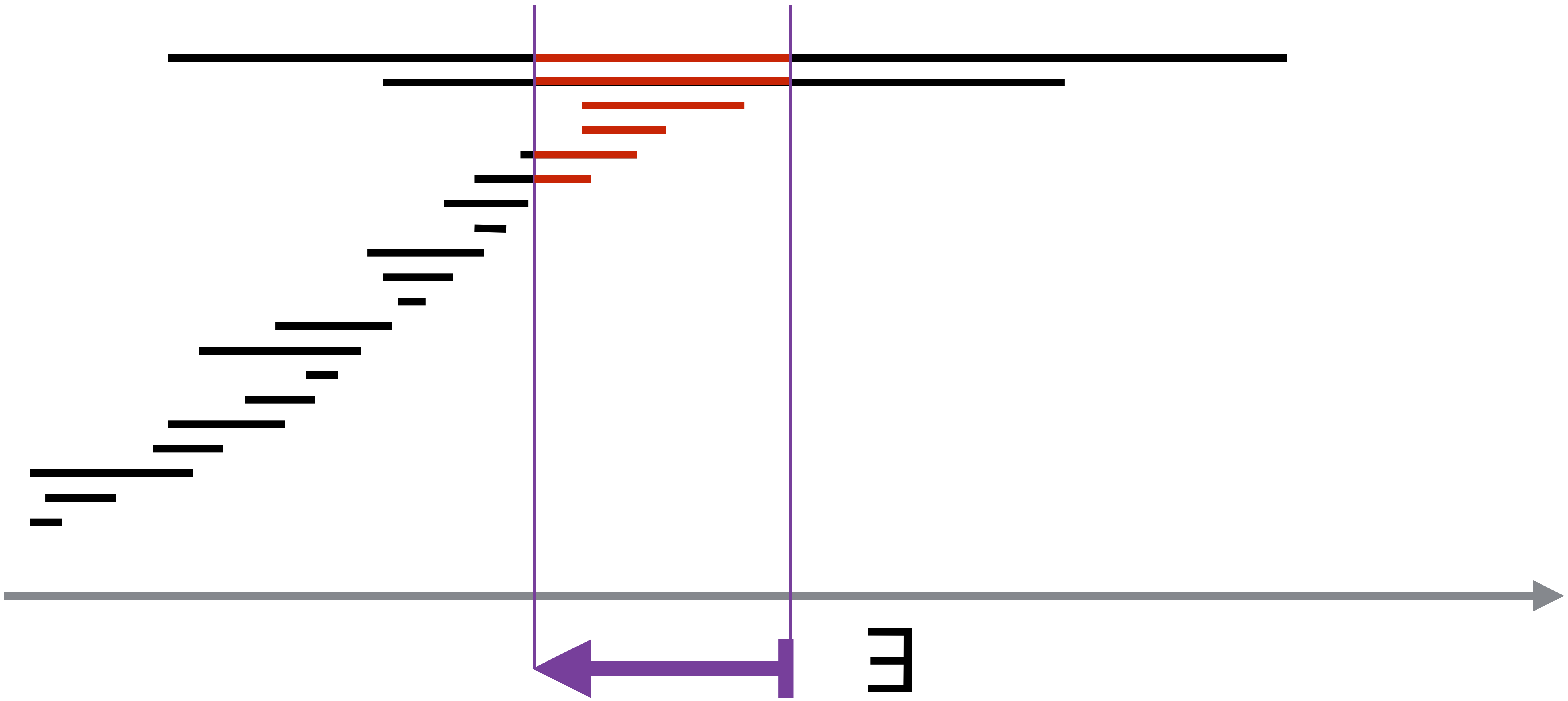
# Elements over intervals



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# Persistent homology

Ambition:

Persistent homology should emerge as the immediate result of developing simplicial homology over the topos of persistent sets.

Simplicial complexes?

Well, quantifying over *sub-complexes* gets very large. Any shortening of any element generates a new sub-object.

(Semi-)simplicial sets replace the quantification by maps.

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# Semi-simplicial persistent sets

Δ category with
objects: [n]
morphisms: strictly increasing maps

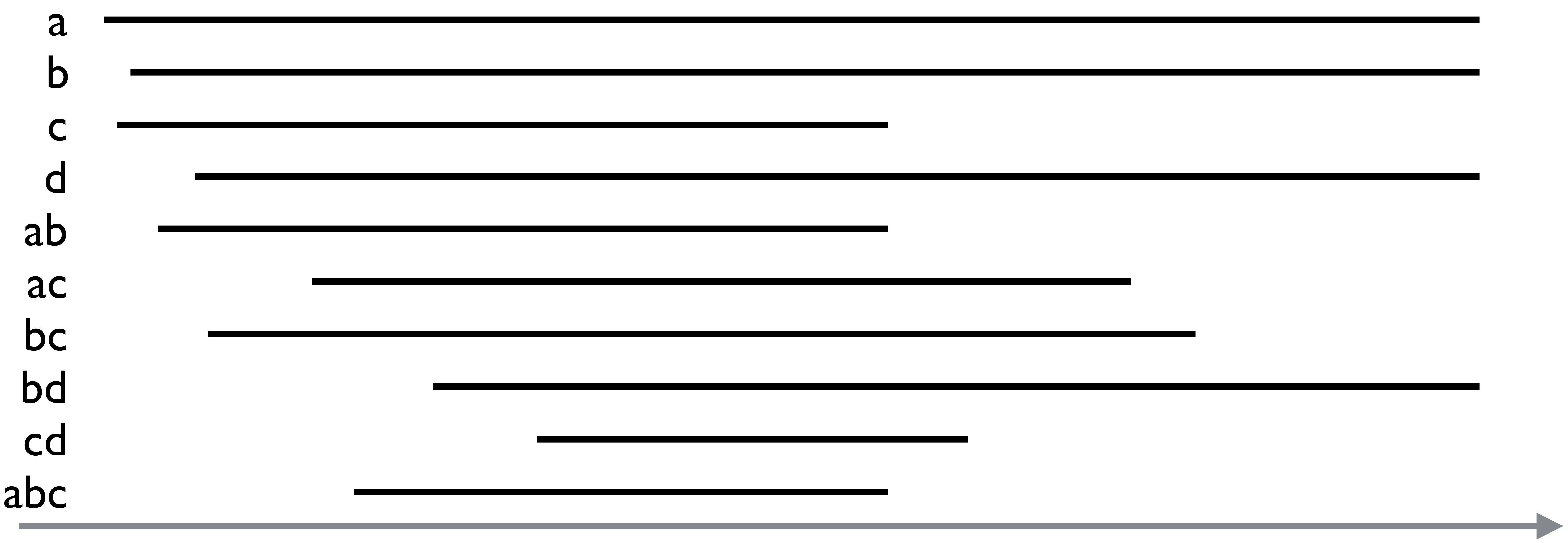
Semi-simplicial set is a presheaf Δ → Set.

Semi-simplicial persistent set is a presheaf Δ → PSet.

Fully specified by a collection of persistent sets of *n-cells* and a collection of *face maps* between them.

We recover persistent homology this way.

# Example



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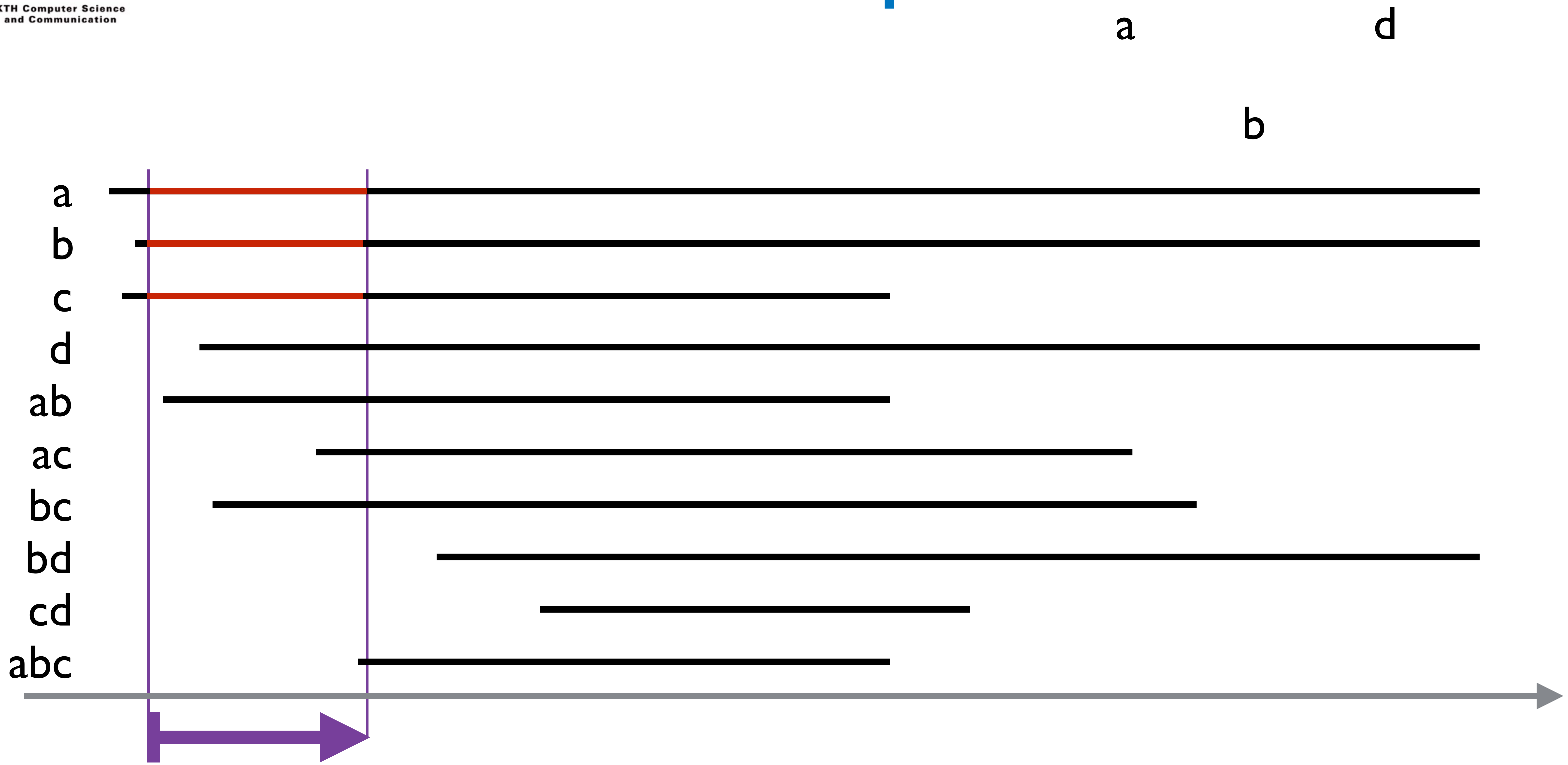

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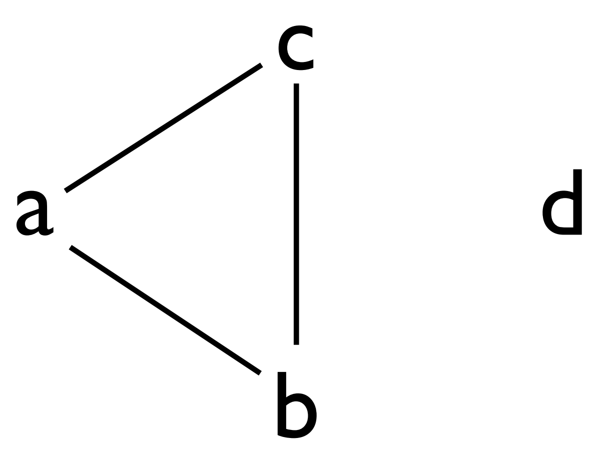


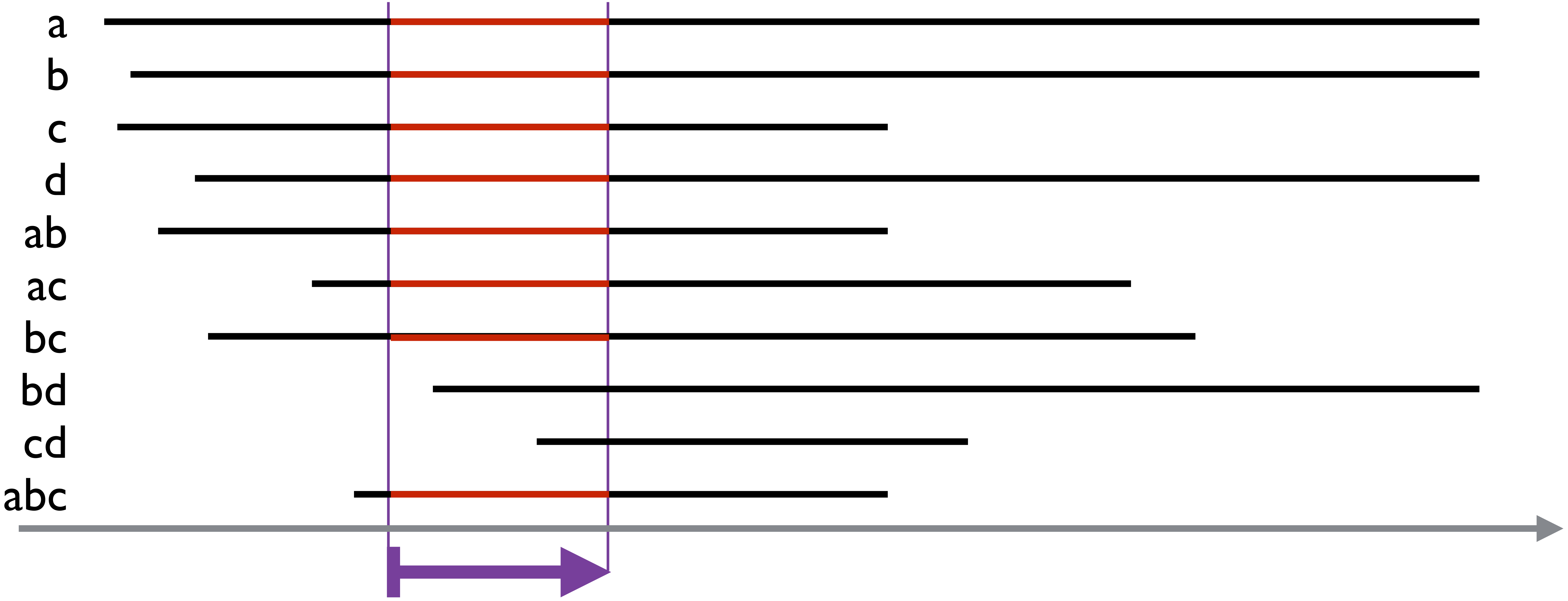

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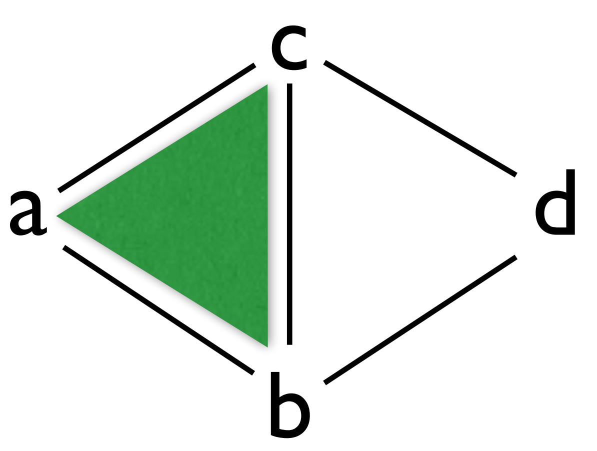


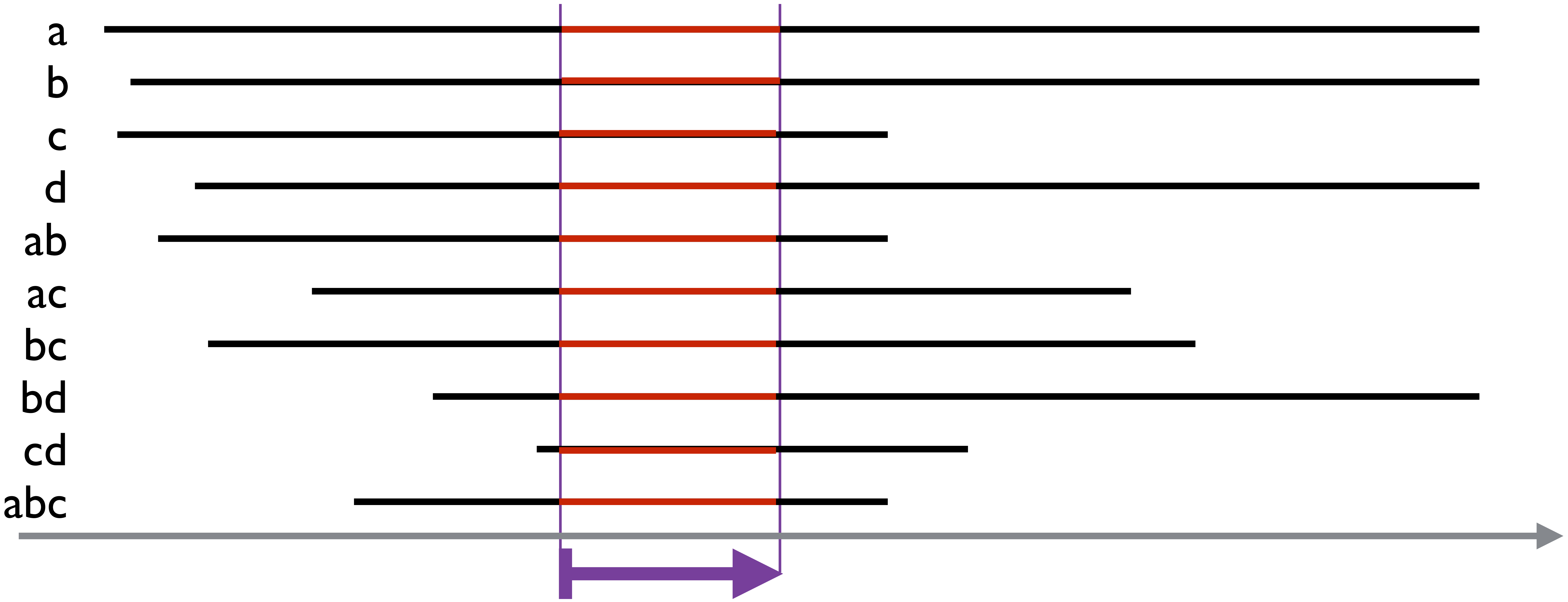

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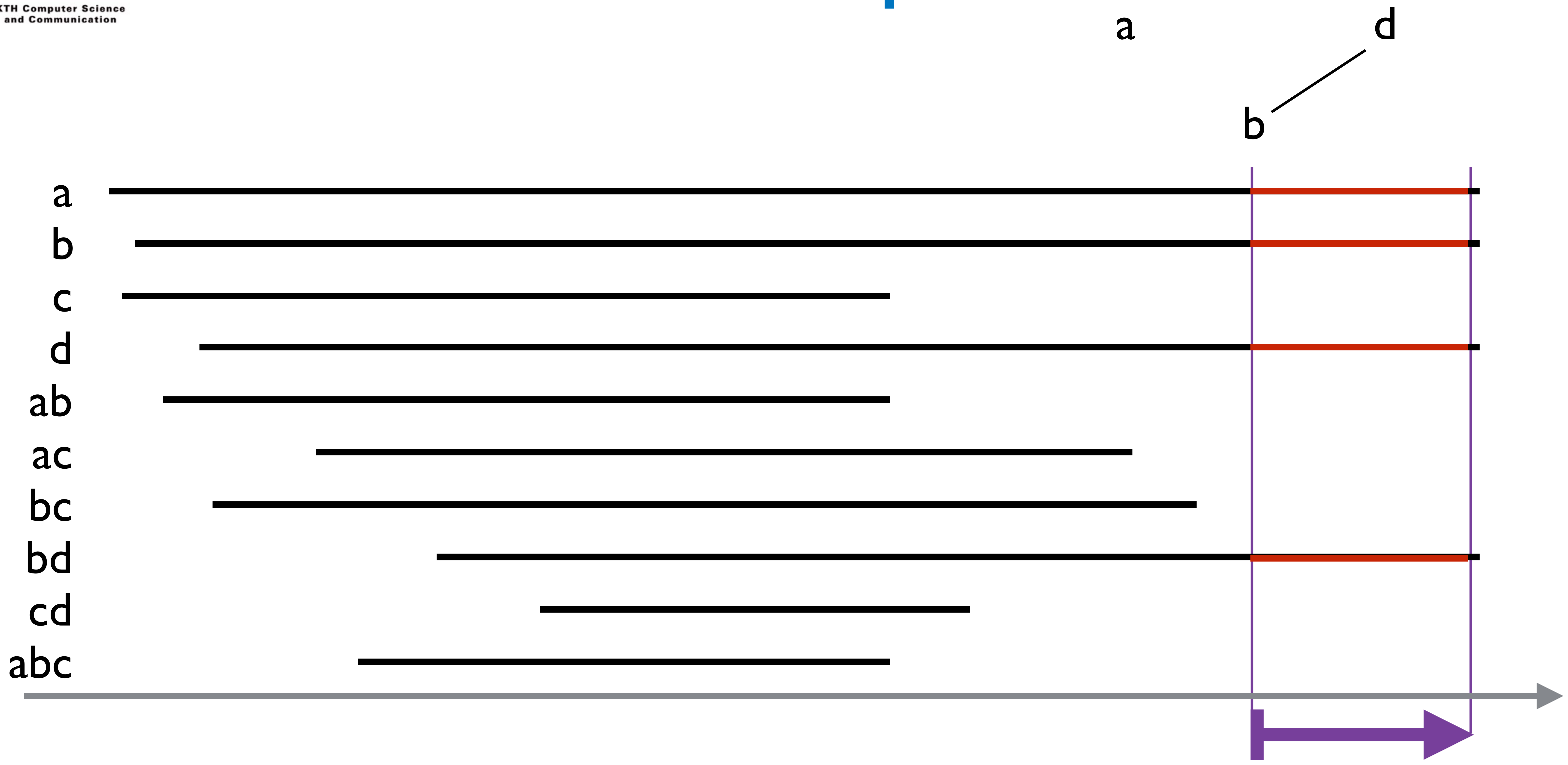

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# Unified theory of persistence

Anything we can prove in generality carries to any shape that has a Heyting algebra.

We have Heyting algebras for:

- Classical persistence
- Zigzag persistence
- 1-critical multidimensional persistence
- 1-critical multidimensional zigzag persistence
- Convex footprint multidimensional persistence
- Circular persistence (aperiodical parts not checked yet)

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# Current directions of research

- Can we prove stability in this setting?
  We build on de Silva-Munch-Patel through constructing a thickening endo-map on the site P.
  This produces a thickening endo-functor on PSet.
- What results can we prove once and for all?
- Can we extract algorithms from this?